Parameter Estimation Project

SIRC Epidemiological model

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Chapter 1

Preparation of the simulations

The SIRC (Susceptible-Infected-Recovered-Carrier) epidemiological model has four state variables:

- S represents the number of susceptible,
- I denotes the number of infected,
- R is the number of recovered,
- C is the number of carrier individuals.

The model expresses that:

- Susceptible individuals might become infected by either infected or carrier individuals.
- The infected individuals are infectious for a while, then they become either recovered or carriers.
- Carriers have no symptoms, although they can infect others.

The dynamical behaviour of the system is described by the following set of differential equations:

$$\begin{cases} \frac{dS}{dt} &= \nu - (\beta I + \epsilon \beta C) S - \mu S \\ \frac{dI}{dt} &= (\beta I + \epsilon \beta C) S - \gamma I - \mu I \\ \frac{dC}{dt} &= \gamma q I - \Gamma C - \mu C \\ \frac{dR}{dt} &= \gamma (1-q) I + \Gamma C - \mu R \end{cases}$$

The parameters to be estimated are:

- ν corresponds to the birth rate,
- μ is the natural mortality of individuals,
- β is the transmission rate from the susceptible state to infected,
- ϵ denotes the transmission rate representing the infections related to carriers,

- γ represents the rate at which infected become recovered,
- Γ represents the rate at which carriers become recovered,
- q is the proportion of infectious individuals turning to carriers,
- $(1 - q)$ is the proportion of infectious individuals turning to recovered.

1.1. Initial values and parameters

Our chosen initial values and parameters are in the **Table 1.1** with total population of 1000 individuals. At the outbreak of an epidemic, the number of susceptible individuals is high, while the already infected is low and the recovered and carriers are zero. We chose the parameters so that they represent a classical epidemic spread model.

Variables	Initial values
S0	990
I0	10
R0	0
C0	0
ν	14
μ	0.013
β	0.0002
ε	1.2
γ	0.05
Γ	0.03
q	0.8
$(1 - q)$	0.2

Table 1.1: The chosen initial values and parameters

The simulation of the SIRC epidemiological model with the initial values and parameters without noise is visible on the **Figure 1.1**.

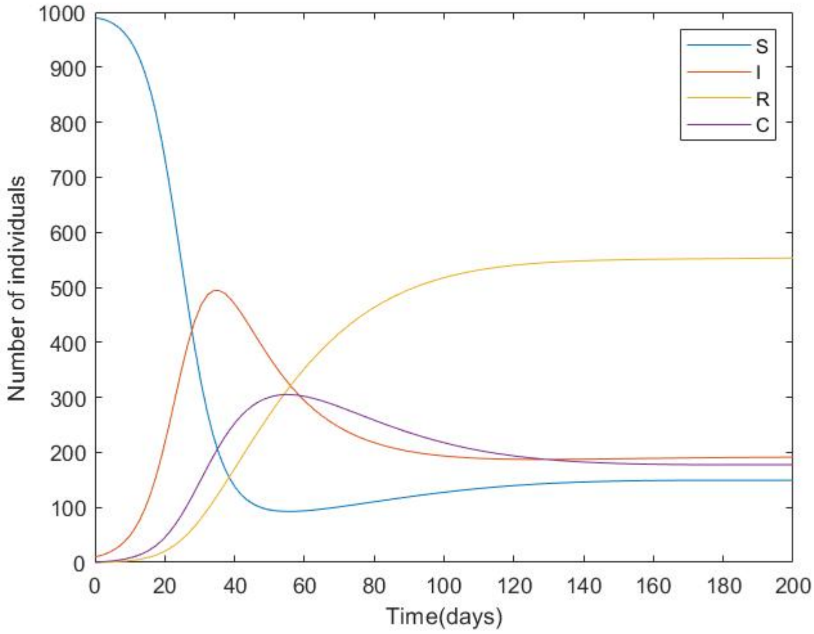


Figure 1.1: The SIRC model without noise.

1.2. Simulation with noise

We also simulated the model with noise to make it more realistic. We chose two noise types, a simple additive Gaussian white noise (**Figure 1.2**) and an autoregressive additive Gaussian noise (**Figure 1.3**), and perturbed the system with them.

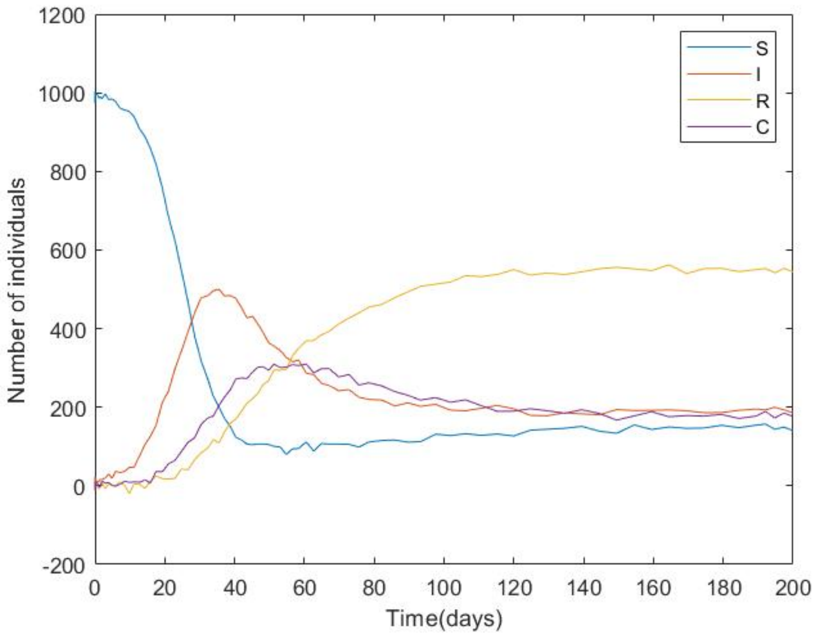


Figure 1.2: The SIRC model with simple additive Gaussian white noise.

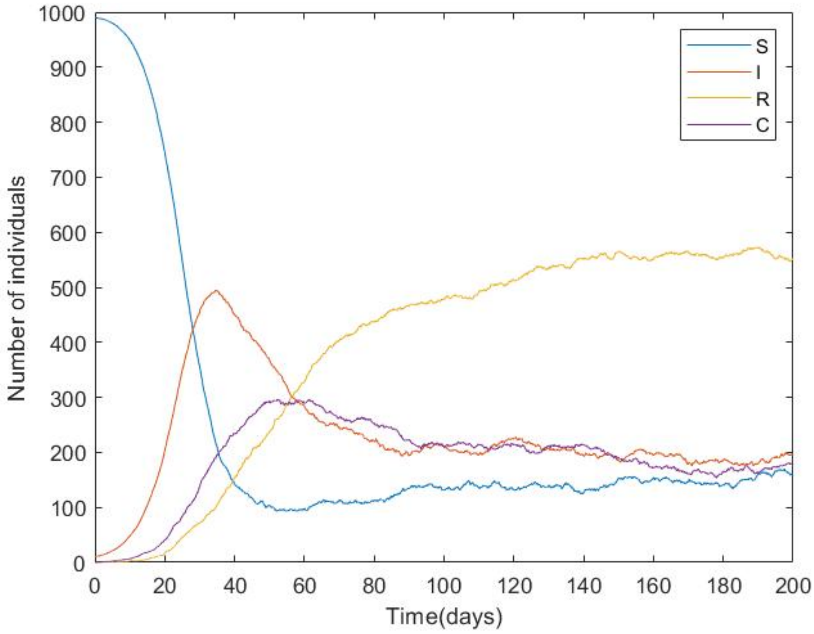


Figure 1.3: The SIRC model with autoregressive additive Gaussian noise.

Chapter 2

Least square estimation

To decide between the systems perturbed with a different type of noise, we calculated their least square estimates and compared them with the estimated parameters of the noise-free system.
For this estimation, we used the discrete-time equivalent of the equations:

$$S(k) = S(k-1) + \nu - (\beta I + \varepsilon \beta C)S - \mu S$$

$$I(k) = I(k-1) + (\beta I + \varepsilon \beta C)S - \gamma I - \mu I$$

$$C(k) = C(k-1) + \gamma q I - \Gamma C - \mu C$$

$$R(k) = R(k-1) + \gamma(1-q)I + \Gamma C - \mu C$$

To simplify the equation system, we merged the equations describing the change of S and I. Moreover, we removed the parentheses so that all the equations contain state variables at most once.
After this simplification, we got the following equation system:

$$S(k) + I(k) = S(k-1) + I(k-1) + \nu - \mu S - (\gamma + \mu)I$$

$$C(k) = C(k-1) + \gamma q I - (\Gamma + \mu)C$$

$$R(k) = R(k-1) + \gamma(1-q)I + (\Gamma - \mu)C$$

Then we used these equations to estimate all parameters of the system with the least square estimation method. Table 2.1 contains the estimated parameters.

Params	Original	Noise-free	Additive Gaussian	Autoregr. add. Gaussian
ν	14	24.3200	26.5132	0.0433
μ	0.013	0.0236	0.0261	0.000041
β	0.0002	0.000353	0.000347	0.000000212
ϵ	1.2	1.118	1.192	6.9
γ	0.05	0.0841	0.0879	0.000175
q	0.8	0.8935	0.8963	0.5871
Γ	0.03	0.115	0.12	0.000151

Table 2.1: Least-square estimation of the parameters

Based on the values above, the LS estimation of the noise-free system and the system perturbed with additive Gaussian white noise resulted in similar parameters. The parameters were higher in both cases than the original ones, however, they had similar order of magnitude.

In contrast, the results of the least square estimation of the system perturbed with autoregressive additive Gaussian noise did not show any similarity with any other parameters. The relation of the parameters was also completely different than in the other cases, thus we decided to use the system perturbed with simple additive Gaussian noise for the following approximation methods.

However, the parameters estimated from the noise-free model and the system perturbed with additive Gaussian noise were not good enough to approximate the original system. The systems built using the estimated parameters show a completely different infection propagation than the original one.

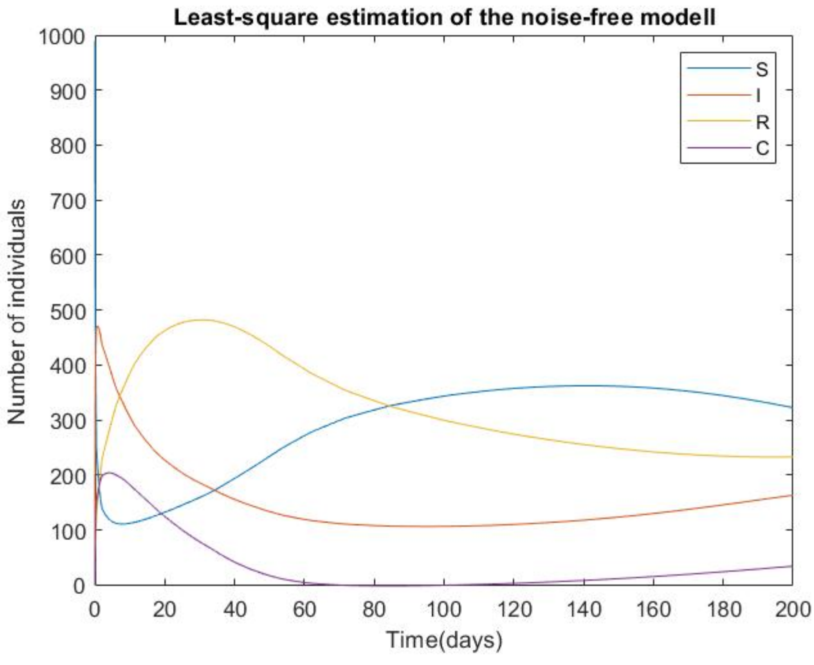


Figure 2.1: Reconstructed model based on the LSQ parameters of the noise-free system

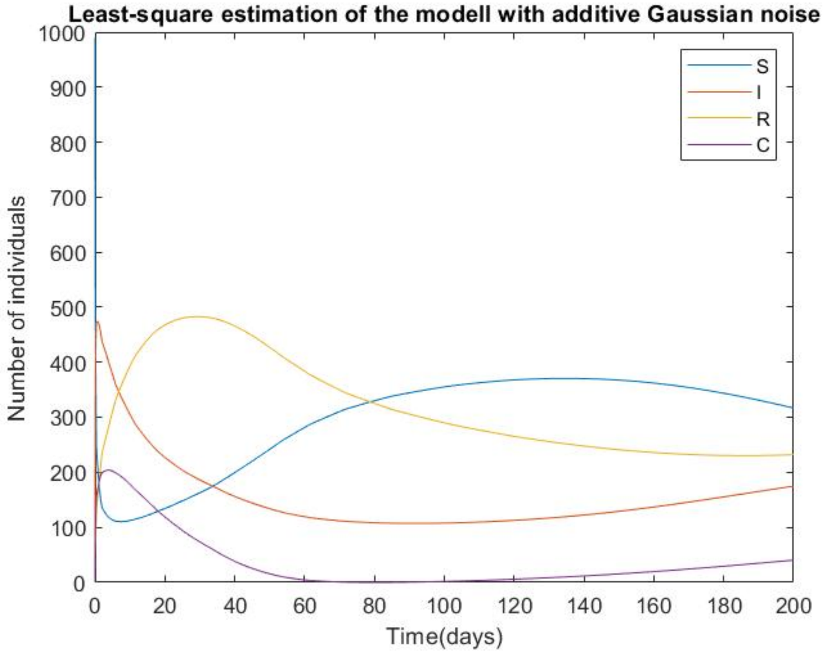


Figure 2.2: Reconstructed model based on the LSQ parameters of the system perturbed with Gaussian white noise

Chapter 3

Instrumental variable method

Since the estimated least-square values were not optimal parameters, we tried to correct them using the instrumental variable method. This algorithm is widely used to identify weakly damped or unstable systems, thus it is a remarkable solution if the approximation errors are coming from some kinds of correlation.

We used the previous least-square parameters computed from the system perturbed with additive Gaussian noise to construct the instrumental variables. With these variables, we computed the corresponding IV estimate and the error for the model.

The instumental variable method resulted in the parameters of Table 3.1.

Params	Original	Additive Gaussian	IV4 estimate
ν	14	26.5132	20.2119
μ	0.013	0.0261	0.0195
β	0.0002	0.000347	0.000381
ϵ	1.2	1.192	2.6898
γ	0.05	0.0879	0.0776
q	0.8	0.8963	0.9473
Γ	0.03	0.12	0.1084

Table 3.1: Comparison of the original the LSQ and the instumental variable estimate

Unfortunately, the instrumental variable method did not improve obviously all the parameters. In particular cases the parameters got closer to the original values indeed, like in case of ν, μ or γ, but other times the estimated values were even further from the original parameters than the least square estimations.

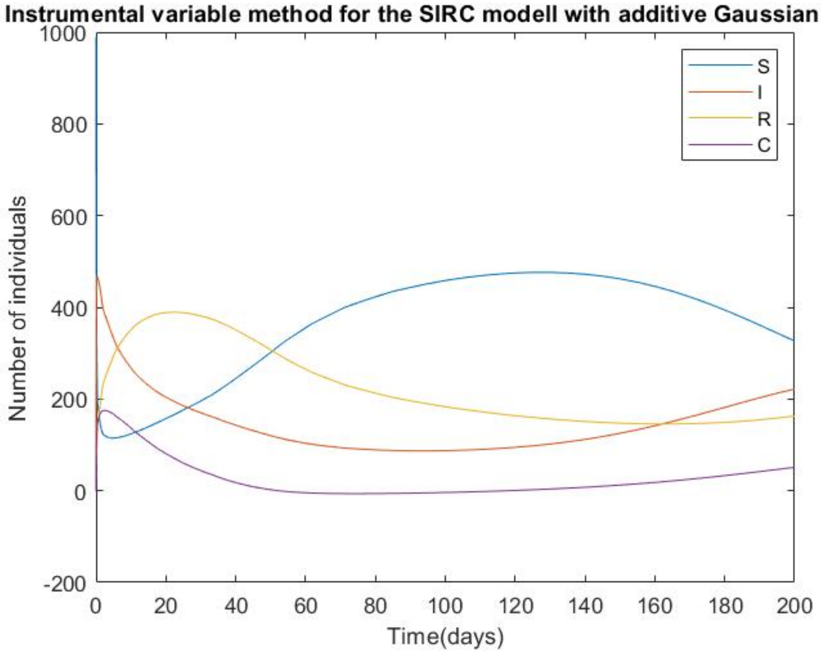


Figure 3.1: Reconstructed model based on the IV4 parameters of the system perturbed with Gaussian white noise

The figure shows that the disease propagation slightly changed compared to the least square estimate, however, it is still not alike to the original one. Probably if we would perform the algorithm multiple times, the result would be more similar to the original, noise-free graph.

Chapter 4

Gradient descent method

Gradient descent method is a widely used optimization algorithm used to find the local minimum of an arbitrary objective function. The idea behind this algorithm is to repeatedly decrease the parameter values by their gradient, thus they get closer and closer to a local minimum value.
To implement this algorithm, first we had to choose an objective function and some initial parameters. We used the gradient descent method to improve the least-square estimations, thus we used the X matrix, the output and the LS parameters of the system perturbed with additive Gaussian noise. We used 0.01 as the learning rate, thus the change of the parameters would be slower and easy to keep track of and we chose the mean squared error as the cost function. We applies the algorithm using one and five steps, since the parameter values were close to optimal. Table 4.1 contains the estimated parameters.

Params	Original	Additive Gaussian	1 step GD	5 step GD
ν	14	26.5132	25.3	4.59e+19
μ	0.013	0.0261	643.7	4.48e+22
β	0.0002	0.000347	0.00726	2.69e+58
ϵ	1.2	1.192	6.97e+5	4.98e+06
γ	0.05	0.0879	0.000549	1.18e+24
q	0.8	0.8963	8.06e-5	5.24e-17
Γ	0.03	0.12	657.35	4.47e+22

Table 4.1: Parameters optimized with the gradient descent method

The estimation was rather unsuccessful, since the parameter values got even further from the optimum. The problem probably comes from the fact that the equations contained a lot of parameters as sums or multiplications, so we could only approximate these. Moreover, even if mean squared error is one of the most common cost functions, it might not be appropriate in this case.

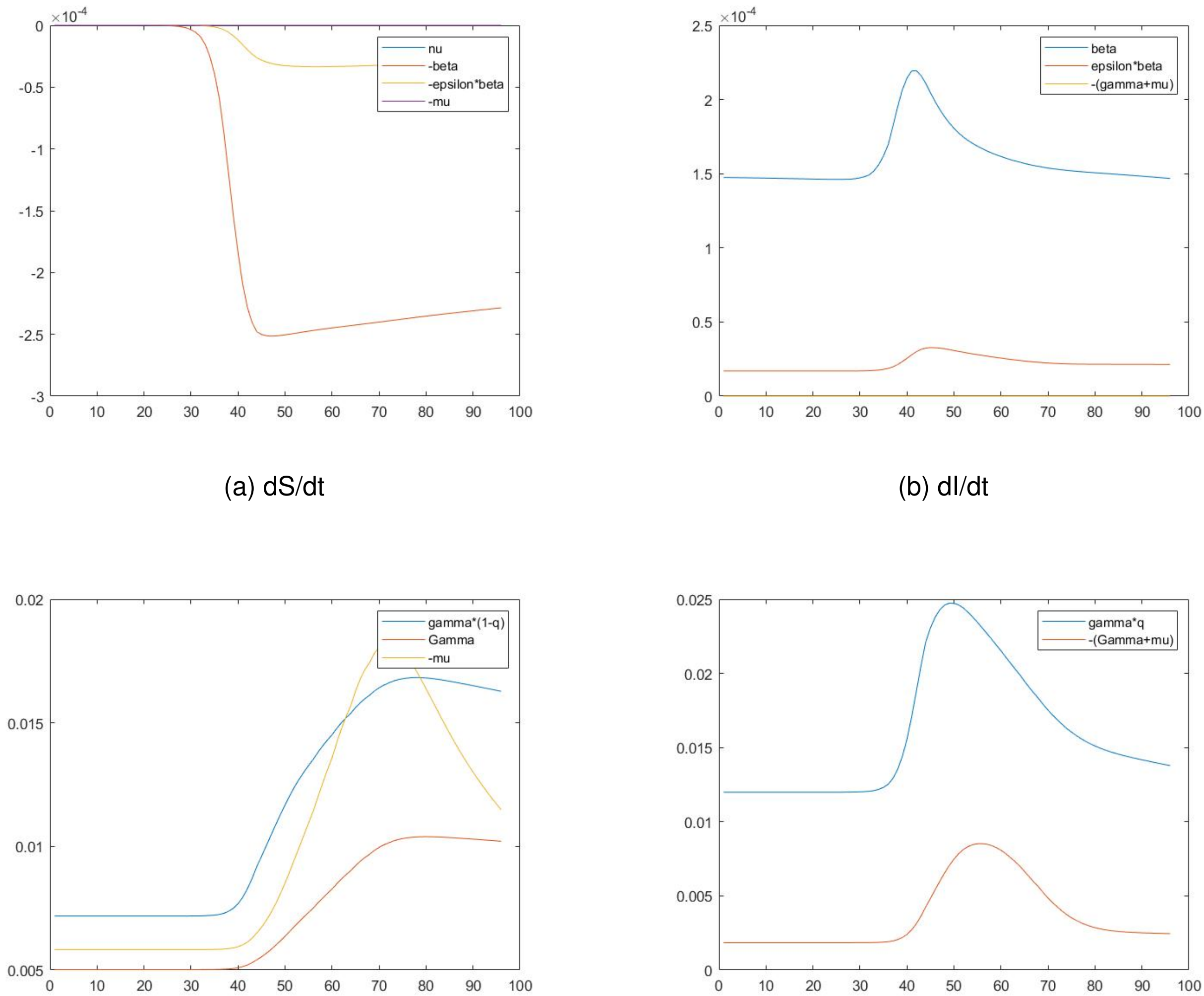
Chapter 5

Recursive Least Square

In the Recursive Least Square (RLS) estimation, the forgetting factor (λ) reduces the influence of old measurement data, so old data has less weight in the parameter estimation over time. This phenomenon is called exponential forgetting. If the value of λ is small enough, then the estimation quickly follows the changes in the parameter meaning that the system forgets quickly, but at the same time, the estimation becomes more sensitive to the noise and possible modeling errors. The opposite is true if the value of λ is close to 1. Recursive least square estimation is useful if we are dealing with a noisy signal, but it is not applicable with systems containing a great amount of noise. The RLS algorithm is appropriate for studying real-time data and in case of an epidemic, data comes continuously. A drawback is that RLS can be stuck in a local minima.

We tried to estimate the parameters with this method in the case of all simulated systems. However, the noise also caused problem for this method. Moreover, if the model is not correctly formulated, problems can appear in the estimation. With the simple additive Gaussian noise, we got approximately the same results as with the noise-free model. For some of the parameters (β, $\gamma(1 - q)$), we got values near to the true values.

The RLS algorithm is mainly used to estimate parameters varying in time. However, our system is time-invariant, meaning that it is not the best method for this particular system.



(a) dS/dt

(b) dI/dt

(c) dR/dt

(d) dC/dt

Figure 5.1: The estimation with RLS